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A radar bomb scoring method

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A RADAR BOMB SCORING METHOD

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ABSTRACT

The major features of a proposed approach to radar bomb scoring are discussed. The method uses observed deviations from desired release conditions as a basis for predicting bomb mean point of impact in the target plane. Circular Error Probable (CEP) is estimated using a noncentral chi-square approximation of the bomb impact distribution. Contributions of bomb scoring system errors (such as errors in radar location) to the apparent delivery system CEP are discussed.

1. Introduction.

We consider the problem of estimating air-to-ground delivery accuracy from radar observed release conditions, rather than impact data. Such procedures can relieve requirements of actually dropping bombs in training or test situations, with corresponding reduction of restrictions on resources such as bomb run location or test time required to complete system evaluations. In what follows, we present an approach which does not require extensive background and training (mathematical, computational, or procedural) on the part of bomb scoring personnel.

Motivation for much of what follows was provided by related circular error probable (CEP) and bomb dispersion analyses for the AN/TPQ-27 radar bombing system which were conducted at the Naval Postgraduate School [1,2]. The proposed procedure suggests use of a computer assisted, radar controlled bombing system, with the AN/TPQ-27 application serving as a prototype example. The proposed method envisions a computer software package for the AN/TPQ-27, or similar system, which could provide a readout of release condition errors or deviations from predetermined release conditions. Through ballistic considerations, these deviations are then translated into range and deflection "aim" errors on the ground. Computation of the estimated CEP is accomplished by means of a model based on the non-central chi-square distribution, where the non-centrality parameter is a function of the computed range and deflection aim errors.

Since in theory, release conditions required to make the mean point of impact (MPI) coincident with a given point on the ground are not unique, it has been suggested that delivery system performance might be evaluated in terms of the envelope of all possible combinations of such conditions. This envelope is in a space of a dozen or more dimensions, counting components of position, velocity and acceleration; it appears to be difficult to determine these envelopes and manipulate them so as to determine median miss distance. As we shall see, it is relatively simple to relate errors in those components to MPI location in the target plane, at least to a reasonable approximation. Since the latter approach seems not only more tractible, but more natural as well, we have abandoned the "envelope" approach. Keene [7] has discussed a process for transforming "small" errors in certain components of delivery conditions, to MPI estimates. In what follows we outline the principle ideas involved. In addition, we discuss effects of errors, such as radar location error, which are outside of the delivery system itself. In section 3, we discuss approaches to extracting such errors from the CEP estimate, so they would not be erroneously charged against the system under test.

2. Transforming Delivery Conditions to the Target Plane.

Except for transient effects such as turbulence, it should be possible to measure quite accurately most of the forces acting on a bomb at the time of its release. To do so, however, would require extensive instrumentation aboard the delivery aircraft and a means of transmitting these measurements to the ground for

analysis. Thus, the most attractive attributes of radar bomb scoring, namely, low cost and flexibility, might be lost in an attempt to obtain all of the measures theoretically possible.

An alternative approach is to base the trajectory estimation on data which can be obtained from a radar, or radars, tracking the delivery aircraft. This greatly simplifies the computational complexity of the MPI estimation problem by limiting the number of parameters to be considered in the equations of bomb motion. We assume the radar can provide information on the coordinate location of the aircraft at time of release, as well as aircraft velocity and acceleration. In addition to tracking information, we assume other "inexpensive" inputs are available from sources outside the radar-aircraft system. Meteorological data and the effects of earth curvature and rotation may be input as standards for the location of the bomb MPI. In addition, bomb parameters, such as drag curves, may be available for the type bomb being dropped.

The MPI may be estimated in either of two ways, which we call "ballistic computation" and "bombing tables." The first of these utilizes ballistic equations to calculate a predicted bomb path and its resultant estimated MPI. Ballistic equations of motion usually involve consideration of aerodynamic forces, variable winds, density and temperature variations, the effects of earth curvature and rotation and bomb parameters such as weight, diameter and configuration. As mentioned above, these parameter values, necessary for solution of the ballistic equations, may be available from sources outside the bomb scoring system. As is pointed out by

McShane, Kelley and Reno [8], computer assistance is required for the numerical solution of such equations.

The complexity of the ballistic computation model may make it too cumbersome for some applications. If so, an alternate approach can be used, which takes advantage of the fact that much of the work in computing trajectories from models of this type has been accomplished and documented in the form of trajectory and bombing tables. The use of these tables provides a quick and computationally simple means of estimating the MPI when the appropriate corrections for existing local conditions are applied to the tabled values. Furthermore, tables are available for nearly all ordnance of interest for a bomb scoring system [9]. The rationale for suggesting use of these tables in precalculating desired bomb release conditions is that this approach is simple (inexpensive) and is widely accepted, in that their use is common practice in combat situations even where there may be critical accuracy requirements.

With either of the two methods of MPI estimation mentioned above (ballistic computation or bombing tables), evaluations of bombing system accuracy can be made for either a pilot controlled or computer controlled mode of bombing. In the former case, the radar measurements of release conditions provide points of entry into the appropriate tables or input values for ballistic computations. The tables provide range as a function of release altitude above the target and release velocity under assumed atmospheric conditions. To these tabled values, corrections due to non-standard

conditions may be applied. With the direction of bomb release available from tracking data, the estimated point of impact may be determined. Comparison of this point with the target location yields the desired estimates of range and deflection components of the estimated MPI. With the ballistic computation method, an MPI is estimated from observed tracking data through a computer solution of the ballistic equations.

It should be noted that for a computer controlled mode of bombing, the scoring system could be physically the same as the control system. Such a configuration is not as desirable in some respects as one using separate physical systems, because with the former, control system errors may not be detected. Also, it is possible to reduce scoring system errors by choosing an advantageous location near the target. Nevertheless, control and scoring with a single physical system might be of interest for reasons of economy. In the case of computer controlled bombing and scoring with a common physical system, the procedure would be somewhat different. Local wind conditions and atmospheric data, target and radar data and weapon ballistics are preset inputs to the control computer. The desired release conditions calculated with these data provide and estimate of the release conditions which would place the expected point of impact on the target. If it can be assumed that deviations from the desired conditions at release are relatively small and can be measured to within small error, only the magnitudes of these deviations need be considered in estimating range and deflection errors. The restriction that these deviations be small

is necessary to insure that both the desired and achieved trajectories are subjected to very nearly the same conditions and forces. The trajectories will then be theoretically nearly identical in shape, so the release deviations may be translated through simple relationships to range and deflection aim errors in the target plane. It would be possible to pre-calculate and tabulate range and deflection components of estimated MPI as a function of deviations in actual release conditions from desired release conditions.

A detailed discussion of how these deviations are translated and combined into MPI estimates is presented and discussed by Keene [7]. Basically, the procedure involves elementary geometrical and positional transformations, and requires the following assumptions:

- the desired release conditions are known from the computer solution of the ballistics model.
- the desired conditions will place the mean point of impact on the target.
- deviations from the desired release conditions are measurable (with small error) and may be output from the bombing system.
- deviations are small.

Since many of the parameters mentioned can be measured fairly precisely, the ultimate accuracy of the predicted impact point will probably depend on the accuracy with which the radar can measure the release conditions: location, velocity, and acceleration. The degree of accuracy obtainable is a function of the radar being used and will vary from one situation to

another. A detailed discussion of radar errors and their determination is beyond the scope of this report. In general, these errors in measurement are related to the aircraft movement relative to the radar as well as the relative positions of the radar and aircraft. The use of multiple radars could reduce tracking errors, at the expense of operating cost and complication of the measurement problem due to the difficulties in calibration and collimation. In addition, the placement of the tracking or "scoring" radar (or radars) relative to the target should be selected so as to reduce tracking errors, near the time of release, as much as possible. The error characteristics of the radar (or radar type) employed may be well known. In a future report we shall discuss how knowledge of radar error characteristics can be used in an adjustment of predicted CEP, which would avoid "charging" radar bomb scoring (tracking system) errors against the delivery system. In the present report, it is assumed such errors are small relative to delivery system and ballistic errors, so significant errors are not caused by ignoring them in the calculated CEP.

3. Effects of Errors on Predicted CEP.

We next discuss how errors, including MPI's offset from the actual target location, affect CEP values. We shall then indicate briefly how estimates of CEP might be adjusted to take such errors and offsets into account. In the foregoing section we described how the MPI associated with a given single bomb run could be estimated from tracking and other data. The amount of offset in a given run of the MPI from the target can be thought

of as bias in the delivery system, for the given run. We desire to relate CEP to bias and ballistic dispersion. Unfortunately, there is not a simple relationship (for even a one-dimensional version of the problem), such as the familiar one involving mean square error, variance, and bias.

If there were no differences in range and deflection variances, due to ballistic errors, and if these components of ballistic error were normally distributed and independent, the ballistic distribution would be the so-called "circular normal distribution" (if the MPI coincided with the target) or "offset circular normal distribution" (if bias was non-zero). In the first case, squared radial error (properly normalized) is chi-square distributed and in the second case it is non-central chi-square distributed. In mathematical terms, suppose $L = \binom{D}{R}$ is the two-dimensional impact location random vector whose first component represents deflection miss and whose second component represents range miss. It is well known [5] that if L is distributed as described above (written "$L \sim N(\mu, \sigma^2 I)$"), then certain quadratic forms in L have non-central chi-square distributions. We write this "$\frac{L'GL}{\sigma^2} \sim \chi'^2_{(k,\lambda)}$," where $\lambda = \mu' G\mu/2\sigma^2$ is the noncentrality parameter of the non-central chi-square distribution with k degrees of freedom. This is the distribution of $L'GL/\sigma^2$ if and only if G is idempotent and of rank k. In particular, with $G = I$, it follows that squared radial miss distance $L'IL = D^2 + R^2$ normalized to variance units, $\frac{L'L}{\sigma^2}$, is distributed $\chi'^2_{(2,\mu'\mu/2\sigma^2)}$. This fact can be used to compute CEP^2 (and hence CEP), which is the median

of this distribution. If $\mu = 0$, i.e. MPI is coincident with the target, the noncentrality is zero and the distribution of $L'L/\sigma^2$ is $\chi^2_{(2)}$, which coincides with the exponential distribution with parameter $\lambda = 1/2$. Hence, by definition,

$$\frac{1}{2} = P[L'L \leq CEP^2]$$

$$= P[\frac{L'L}{\sigma^2} \leq \frac{CEP^2}{\sigma^2}] = 1 - e^{-\frac{1}{2}CEP^2/\sigma^2}$$

which in turn implies the familiar formula

$$CEP = \sqrt{2 \ell n 2}\, \sigma \approx 1.1774\, \sigma. \tag{1}$$

If $\mu \neq 0$, the median of the non-central chi-square distribution corresponds to CEP^2. Unfortunately, this median cannot be expressed in simple closed form as before. We discuss below, in section 4, numerical methods of approximating this value. Moreover, in practice it is unlikely that the components of ballistic dispersion have equal variances. Suppose $L \sim N\left(\begin{pmatrix}\mu_D\\ \mu_R\end{pmatrix}, \begin{pmatrix}\sigma_D^2 & 0\\ 0 & \sigma_R^2\end{pmatrix}\right)$.
Then $\frac{D^2}{\sigma_D^2} + \frac{R^2}{\sigma_R^2}$ is distributed as $\chi'^2_{(2,\lambda)}$, where $\lambda = \frac{1}{2}\left[\frac{\mu_D^2}{\sigma_D^2} + \frac{\mu_R^2}{\sigma_R^2}\right]$.
Unfortunately, with $\sigma_D^2 \neq \sigma_R^2$, squared radial miss distance, $D^2 + R^2$, can no longer be normalized to a non-central chi-square random variable; rather, various approximations and numerical computations may be used to approximate the value of CEP in this case, as described below in section 4.

The point of the preceding discussion is that bias (i.e. MPI offset from the target) affects the distribution of squared radial miss distance through its non-centrality term, at least to

a good approximation. We shall refer to the bias (fixed offset μ) as a fixed effect, and contrast it with random effects, as follows. Suppose that for a certain bomb run situation the MPI is μ, and for a given bomb run in the situation the actual aimpoint is an unobservable random value A, where $A \sim N(\mu, \Sigma_A)$. Since A is a random variable, aimpoint offset from μ is called a random effect. Now if a bomb were actually released on this run, the location of impact L, would depend on the (unobservable) value of A for that run, as well as ballistic error.

We now give a discussion which arrives at the principle that fixed and random effects may affect CEP in different ways, analogous to their impact on λ. For fixed effects, such as delivery system radar location survey error, delivery system radar alignment error and drag curve error, the effect may show us as a fixed bias effect, that is, an effect upon the mean μ. For random effects, that is, error components that would theoretically vary randomly from one bomb run to another conducted under fixed conditions in a given situation, the effects enter through additive variance factors $(\Sigma_A + \Sigma_L + \ldots)$.

Suppose the conditional distribution (ballistic distribution) of L for given A is $N(A, \Sigma_L)$, where Σ_L represents parameters of ballistic dispersion in deflection and range; for example,

$\Sigma_L = \begin{pmatrix} \sigma_D^2 & 0 \\ 0 & \sigma_R^2 \end{pmatrix}$ in a model described above. The unconditional distribution of L, obtained by integration of the conditional density with respect to the aimpoint density, is $N(\mu, \Sigma_L + \Sigma_A)$. Thus $L'GL \sim \chi'^2_{(k,\ \lambda)}$ with $k = \text{rank}\ (G)$ and $\lambda = \frac{1}{2}\mu' G \mu$ if and only if $G(\Sigma_L + \Sigma_A)$ is idempotent. In particular, with $G = (\Sigma_L + \Sigma_A)^{-1}$,

$L'(\Sigma_L+\Sigma_A)^{-1}L \sim \chi'^2_{(2,\lambda)}$, with $\lambda = \frac{1}{2}\mu'(\Sigma_L+\Sigma_A)^{-1}\mu$. While it is the squared radial miss distance $L'L$ whose median we seek (we discuss this below), rather than that of $L'(\Sigma_{L}+\Sigma_A)^{-1}L$, consideration of the latter and its distribution provides insight into how fixed and random effects may also affect CEP. Note, for example, that the random effect A affects the non-centrality parameter λ above through the inverse total variance $(\Sigma_L+\Sigma_A)^{-1}$. Recall the fixed effect, μ, affects λ through a quadratic function of the components of μ.

The significance of the preceding discussion for the radar bombscoring problem is that the procedure used in calculating CEP should account for detected foxed effect errors through the estimated μ and for known random effect errors through the accumulated variance-covariance matrix $\Sigma_A+\Sigma_L+\ldots$. Depending on the configuration of the scoring system relative to the delivery system, some of these errors may not contribute to the CEP calculated, even if they are detected or are known in advance. For example, suppose the scoring system is the same as that used for delivery. Then any survey in radar location, radar alignment error, or wind condition error will affect delivery and scoring in identically the same ways, so nothing would be gained in "correcting" delivery and scoring. Rather, in this configuration, only deviation μ of measured and desired drop conditions are used, together with the ballistic variance-covariance, Σ_L. On the other hand, if the delivery system were physically different from the scoring system, wind

condition error would affect both systems the same (and hence "correction" would not be useful), but survey error in locating, say the scoring radar, as well as scoring radar tracking error, would inflate calculated CEP (and thus correction for these errors should be made, if detected, by changing μ appropriately).

There is yet another source of error affecting calculated CEP's: that involved in estimating the MPI, μ. These errors would be due to scoring system errors, such as scoring radar tracking errors. At present it appears that adjusting calculated CEP so that it does not include such scoring system errors (i.e. "discounting" calculated CEP so the delivery system is not incorrectly "charged" with scoring system random effect errors) requires further theoretical development. We assume the scoring system has accuracy characteristics, when properly located near the drop point, such that serious error in calculated CEP does not result from ignoring these errors.

In summary, using our radar bombscoring approach would involve determining inputs μ and Σ to a CEP computation procedure to be discussed in the next section. The vector μ is estimated using output from the scoring system as discussed above. The matrix Σ is the sum of known variance-covariance components due to ballistic error, and delivery system random effects (if any).

4. Computation of CEP Measures.

Mathematically, our problem is to compute the median of the random variable $L'L$, given L has known bivariate normal distribution, $L \sim N(\mu,\Sigma)$. This is a well known "damage" or "coverage" problem, and a large literature exists in connection with it. (See,

for example, the reference list of Johnson and Katz [6].) This problem amounts to finding a radius such that integration of the general bivariate normal distribution over an offset circle of that radius gives value 1/2. Computer routines for numerically approximating such integrals have been published, for example, by DiDonato and Jarnagin [3]. Tables of radius values for selected μ and Σ have also been published [4]. These tables, or an extension of them, might prove adequate for the CEP calculation problem. In what follows, we discuss an alternate approach. We consider first a special case (the non-central χ^2) then the more general case.

If Σ were of the form $\sigma^2 I$ (the offset circular normal situation), then as mentioned above, calculating CEP amounts to finding (or approximating) the median of a non-central χ^2 distribution. Again, tables are available [5] or may be developed giving median values as a function of σ^2 and λ or, perhaps more naturally, as a function of σ^2 and μ. Alternatively, approximate analytical expressions have been developed (see Johnson and Katz [6], for example); a simple approximate median is given by

$$\text{CEP} \approx 1.1774\,\sigma\,(1+.5000\,\lambda+.0817\,\lambda^2) \; . \qquad (2)$$

This expression can be viewed as a modification of the CEP formula (1) used for the non-offset circular normal case. Indeed, with $\lambda = 0$ the offset circular normal situation specializes to the circular normal; formula (2) specializes to formula (1) as should be the case.

It is not difficult to develop a numerical procedure for this two degrees of freedom case which is simple enough to be implemented with a modestly sophisticated calculator. It is well known [5] that the non-central χ^2 distribution can be expressed as a mixture of central χ^2 distributions. Let G_ν denote the cumulative distribution function (CDF) for the χ^2 distribution with ν degrees of freedom. Then the $\chi'^2_{(2,\lambda)}$ CDF is given by

$$F(z) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} G_{2+2k}(z). \tag{3}$$

Repeated application of integration by parts to the integral representation of G_{2+2k} yields the representation

$$G_{2+2k}(z) = 1 - \sum_{j=0}^{k} (\tfrac{z}{2})^j \, e^{-z/2}/j! \tag{4}$$

Replacing G in (3) by this representation, one obtains

$$F(z) = 1 - \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\lambda^k e^{-\lambda}}{k!} \cdot \frac{(\frac{z}{2})^j e^{-z/2}}{j!} . \tag{5}$$

The computation problem can now be stated as follows: find $\eta > 0$ such that

$$e^{-\lambda-\eta} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{j=0}^{k} \frac{\eta^j}{j!} = \frac{1}{2} \tag{6}$$

or (changing order of summation),

$$e^{-\lambda-\eta} \sum_{j=1}^{\infty} \frac{\eta^j}{j!} \sum_{k=0}^{j-1} \frac{\lambda^k}{k!} = \frac{1}{2} . \tag{7}$$

CEP is then given by

$$CEP = \sigma\sqrt{2\eta} \tag{8}$$

A search routine can easily be given to find η for any given $\lambda > 0$, and consequently to give CEP for any $\sigma > 0$, using (8). Such a simple expansion can be obtained for any non-central chi-square distribution with even degrees of freedom; otherwise the expansion would be such that calculation with a computer might be required.

For the non-circular case, $L'L$ cannot be simply normalized to a χ'^2 random variable. We now propose an alternative to the computer solutions mentioned at the beginning of this section. The approach is to approximate the distribution of $L'L$ with that of a function of a certain non-central chi-square distributed random variable. This procedure is similar to that suggested by Pearson [10] for approximating non-central chi-square random variable in terms of functions of central χ^2 random variables. In it, parameters in the approximating function are fitted by moments. Let us suppose, then, that

$$L'L/c \mathrel{\dot{\sim}} \chi'^2_{(\nu,\lambda)} \tag{9}$$

where "$\dot{\sim}$" means "approximately distributed as," the "scale factor" c and "degrees of freedom" ν are parameters whose values must be determined, and where the non-centrality λ is of the form $\lambda = \frac{1}{2c}\mu'\mu$. This form for non-centrality is suggested by the formula for the offset circular normal case.

Assume $L \sim N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\right)$, so $L'L = D^2 + R^2$ and $D^2/\sigma_1^2 \sim \chi'^2_{(1,\mu_1^2/2\sigma_1^2)}$ and $R^2/\sigma_2^2 \sim \chi'^2_{(1,\mu_2^2/2\sigma_2^2)}$. It follows that

$$E(L'L) = \sigma_1^2 E(D^2/\sigma_1^2) + \sigma_2^2 E(R^2/\sigma_2^2)$$

$$= \sigma_1^2(1+\mu_1^2/2\sigma_1^2) + \sigma_2^2(1+\mu_2^2/2\sigma_2^2), \tag{10}$$

and

$$V(L'L) = \sigma_1^4 V(D^2/\sigma_1^2) + \sigma_2^4 V(R^2/\sigma_2^2)$$

$$= 2\sigma_1^4(1+\mu_1^2/\sigma_1^2) + 2\sigma_2^4(1+\mu_2^2/\sigma_2^2). \qquad (11)$$

On the other hand, by assumption (9) it follows that

$$E(L'L) \doteq c(\nu+\lambda)$$

$$= c(\nu + \frac{1}{2c}(\mu_1^2+\mu_2^2)) \qquad (12)$$

$$V(L'L) \doteq 2C^2(\nu+2\lambda)$$

$$= 2C^2(\nu + \frac{1}{C}(\mu_1^2+\mu_2^2)). \qquad (13)$$

Equating expression (12) with (10), and (13) with (11), and solving simultaneously for c and ν, we obtain

$$c = \frac{\sigma_1^2(\sigma_1^2+\mu_1^2) + \sigma_2^2(\sigma_2^2+\mu_2^2)}{(\sigma_1^2+\mu_1^2) + (\sigma_2^2+\mu_2^2)} \qquad (14)$$

$$\nu = 1 + \frac{\sigma_1^2(\sigma_2^2+\mu_2^2) + \sigma_2^2(\sigma_1^2+\mu_1^2)}{\sigma_1^2(\sigma_1^2+\mu_1^2) + \sigma_2^2(\sigma_2^2+\mu_2^2)} \qquad (15)$$

In practice, for associated CEP computation, it might be more tractible to round the value of ν computed in (15) to the nearest integer value ν^*, and to take the corresponding c value, $c = (\sigma_1^2+\sigma_2^2)/\nu^*$. For a number of example μ and Σ values we examined, ν^* ranged between 1 and 3, which appears quite reasonable in comparison with the offset circular case. Indeed, acceptable accuracy might be achieved in many applications by taking $\nu = 2$ and $c = (\sigma_1^2+\sigma_2^2)/2$. This choice would allow use of the simple calculation approach of equations (6) and (7). Note

that in the offset circular case (i.e. $\sigma_1^2 = \sigma_2^2$), use of formulas (14) and (15) would give the correct distribution of $L'L$, given earlier: With $c = \sigma_1^2$ and $\nu = 2$, the distribution of $L'L/C$ is exactly $\chi'^2_{(2,\mu'\mu/2c)}$.

5. Summary.

The approach we have described consists of using radar track, and other information, depending on the systems used, to estimate the MPI which would result if bombs were actually released. This vector, μ, together with known ballistic and delivery system variance/covariance, Σ, can then be used in estimating the CEP associated with these values. The computation is based on approximating the distribution of squared radial miss distance with a non-central chi-square distribution whose parameters are functions of the estimated μ and known Σ. Several approaches to computation of medians of such non-central chi-square distributions have been mentioned, and one procedure for use with the "even degrees of freedom" case was presented.

The method of estimation of the MPI for a given bomb run depends upon the systems being used for delivery and scoring. The combinations we discussed are indicated in the following diagram with the cells marked "x."

System Configuration		Computation Method	
Delivery	Scoring	Ballistic Computation	Bombing Tables
Pilot controlled	Separate	X	X
System controlled	Separate	X	
System controlled	Common	X	

The roles of various potential error sources may be different in the "separate" scoring system case from that in the "common" system. Example fixed, random and unaccounted effects are shown below.

	Fixed Effect	Random Effect	Unaccounted
Common delivery and scoring system	Pilot error Delivery system error (such as not dropping at proper time) System inability to achieve exactly desired drop condition	Ballistic error	Air density profile Drag curve Radar survey error Radar track error
Separate scoring system	Survey error Delivery System Tracking Error	Scoring system tracking error	Errors in drop condition inputs (winds, drag, etc.)

Several problems remain with respect to details of implementing the approach suggested herein, and effort should be directed toward their solution. First, what is the best way to calculate (or approximate) the median of $L'L$ in the non-circular offset normal case, and what accuracies can be expected with each candidate method? Second, how should calculated CEP's be "discounted" to remove errors and variance components which are due to the scoring system, rather than the delivery system? Third, what is the best way to combine CEP estimates made in individual runs into a single CEP measure for a "sortie?" We plan to consider these and related questions in future efforts.

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